J. Cent. South Univ. Technol. (2007)05−0708−05
DOI: 10.1007/s11771−007−0135−8

Springer

Mapping analysis of vibrating fundamental frequency for simple-supported elastic rectangle-plates with concentrated mass

QI Hong-yuan(齐红元), WANG Shuang-xin(王爽心), ZHU Heng-jun(朱衡君)

(School of Mechanical and Electric Control Engineering, Beijing Jiaotong University, Beijing 100044, China)

Abstract: By conformal mapping theory, a trigonometric interpolation method between odd and even sequences in rectangle boundary region was provided, and the conformal mapping function of rectangle-plate with arc radius between complicated region and unite dish region was carried out. Aiming at calculating the vibrating fundamental frequency of special-shaped, elastic simple-supported rectangle-plates, in the in-plane state of constant stress, the vibration function of this complicated plate was depicted by unit dish region. The coefficient of fundamental frequency was calculated. Whilst, taking simple-supported elastic rectangle-plates with arc radius as an example, the effects on fundamental frequency caused by the concentrated mass and position, the ratio of the length to width of rectangle, as well as the coefficient of constant in-plane stress were analyzed respectively.

Key words: conformal mapping; elastic simple-supported plates; vibration; fundamental frequency; mode; method of trigonometric interpolation

1 Introduction

In the in-plane state of constant stress, for the region and geometrical boundary of complicated elastic plates with concentrated mass, such as non-round region like non-square shape with arc radius and so on, it is too difficult to set up the mathematical analysis of fundamental frequency of elastic plates region. The horizontal analysis of vibrating fundamental frequency can therefore not be achieved[1−2]. The key point is that no mathematical model can be determined in the plate region and its geometrical boundary to analyze vibrating differential equation. At present, there is not systematical numerical-value analysis method, the solution is thus limited in using finite element method and finite differential method to do approximate calculation[3−4]. During the engineering design of complicated circuit board PCB (rectangle-plate region with arc radius) with concentrated substance of electrical components, the calculation of natural frequency of plate can prevent resonance from electrical components and underhead crack, so as to reduce the injury to PCB system and improve its dynamic performance[5]. Therefore the solutions are truly useful in actual practice. Now, with the help of conformal mapping theory, using given mapping function to convert typical region into simple region[6−8], which can simplify the complicated region into simple region, and can provide a feasible analysis method to complicated methermatical modeling, as the result, some engineering problems can be solved. Conformal mapping theory has been applied in stress machining, hydromechanics and electromagnetic engineering fields[9−11]. When modeling special-shaped product extrusion, the complicated die cavity and deforming region can be changed into axis-symmetry to do analysis[12]. In the in-plane state of constant stress, in order to calculate the fundamental frequency of elastic simple-supported rectangle-plates region with concentrated mass, a trigonometric interpolation method was provided here, by conformal mapping function, the mutual conversion between rectangle-plates region and unit dish region can be realized. Meanwhile, the analysis was completed by taking the vibrating differential function and fundamental frequency of plate region with arc radius as examples.

2 Differential function of vibration

As shown in Fig.1, for the complicated elastic simple-supported rectangle-plate with arc radius R, whose mass is m_0, assuming concentrated mass is m_p at the point $p(x_p, y_p)$, from small-amplitude vibrating differential function[13] in the in-plane state of constant stress S, we can get

$$G\nabla^4 D(x,y,t) - S\nabla^2 D(x,y,t) +$$

$$\left[1+\lambda A\delta\left(x-x_p\right)\delta\left(y-y_p\right)\right]\rho_{\rm r} h\left(\frac{\partial^2 D}{\partial t^2}\right)=0 \qquad (1)$$

Received date: 2007−01−20; **Accepted date**: 2007−03−23
Corresponding author: QI Hong-yuan, PhD; Tel: +86-13611009942; E-mail: hyqi@bjtu.edu.cn

where $D(x, y, t)$ is deforming deflection of elastic plate; G is bending rigidity of plate; $\lambda=m_p/m_0$ expresses the coefficient of substance ratio; A is the area of plate, ρ_r is the density of plate material; m_0 is the plate mass; m_p is the concentrated mass on the plate; S is the constant in-plane stress; h represents the thickness of the plate; δ is Dirac function. When calculating the mode, Eqn.(1) can be transferred as follows:

$$G\nabla^4 D - S\nabla^2 D - \left[1+\lambda A\delta\left(x-x_p\right)\delta\left(y-y_p\right)\right]\cdot \rho_{\mathrm{r}} h\omega^2 D = 0 \quad (2)$$

where ω is the circular fundamental frequency.

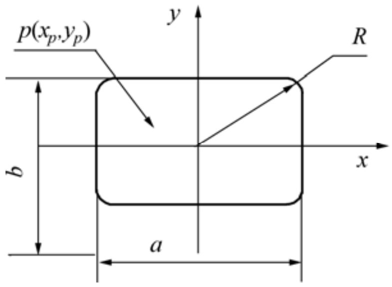

Fig.1 Elastic plate region with m_p

Eqn.(2) is used to do mode calculation for simple plate region, it can be changed into the following integration function to ask for extreme value.

$$f(D)=\iint_A\left\{G\left(\nabla^2 D\right)^2+S\left[\left(\frac{\partial D}{\partial x}\right)^2+\left(\frac{\partial D}{\partial y}\right)^2\right]-\left[1+\lambda A\delta\left(x-x_p\right)\delta\left(y-y_p\right)\right]\rho_{\mathrm{r}} h\omega^2 D^2\right\}\mathrm{d}x\mathrm{d}y \quad (3)$$

But for complicated region problems, it is very difficult to calculate by mathematics methods because the variables cannot be separated. Here, the complicated region $\boldsymbol{W}$ with arc radius can be changed into unit dish region $\boldsymbol{\zeta}(u, v)$, so Eqn.(3) in complex number region can be simply described by the available real number.

$$f(D)=\iint_A\left\{G\left(\frac{\partial^2 D}{\partial\zeta\partial\bar{\zeta}}\right)^2\frac{1}{\|\boldsymbol{W}'\|^2}+4S\left(\frac{\partial D}{\partial\zeta}\right)\left(\frac{\partial D}{\partial\bar{\zeta}}\right)\frac{1}{\|\boldsymbol{W}'\|}-\left[1+\lambda A\delta\left(\mathrm{Re}\boldsymbol{W}\left(\bar{\zeta}\right)-\mathrm{Re}\boldsymbol{W}\left(\zeta_p\right)\right)\delta\left(\mathrm{Im}\boldsymbol{W}\left(\zeta\right)-\mathrm{Im}\boldsymbol{W}\left(\zeta_p\right)\right)\right]\cdot\rho_{\mathrm{r}} h\omega^2 D^2\right\}\|\boldsymbol{W}'\|\mathrm{d}u\mathrm{d}v \quad (4)$$

When the plate's boundary is simple-supported, its boundary condition can be described by

$$D\left(\zeta,\bar{\zeta}\right)\Big|_{|\bar{\zeta}|=|\zeta|=1}=0 \quad (5)$$

The function which satisfies boundary conditions Eqn.(5) can be approximately expressed as

$$D\left(\zeta,\bar{\zeta}\right)=\sum_{n=1}^{N} d_n\left[1-\left(\zeta\bar{\zeta}\right)^n\right] \quad (6)$$

where d_n is the undefined coefficient.

Put Eqn.(6) into Eqn.(4), and ask $f(D)$ for extreme value

$$\frac{\partial f(D)}{\partial d_n}=0 \quad (7)$$

Thus, d_n and the value of fundamental frequency can be obtained.

3 Conformal mapping of complicated region

For the analysis of vibrating differential function and the plate vibrating fundamental frequency, it is the critical point whether elastic plate region and geometrical boundary can be described by a function systemically. Here, by complex conformal mapping theory[14], we used the following complex polynomial function to describe the geometrical region of complicated plate.

$$\boldsymbol{W}=\sum_{n=0}^{\infty} c_n\boldsymbol{\zeta}^n,\quad \boldsymbol{\zeta}=\rho\exp(\mathrm{i}\theta) \quad (8)$$

In polynomial Eqn.(8), $c_n=a_n+\mathrm{i}b_n$ is complex coefficient, mapping function $\boldsymbol{W}$ should satisfy the following boundary conditions:

$$\boldsymbol{W}=f\left(\zeta\right)\big|_{\zeta=0}=0 \quad (9)$$

By Eqn.(9), Eqn.(8) can be described by complex trigonometric function as follows:

$$\begin{cases} x(\rho,\theta)=\sum_{n=1}^{\infty} c_n\rho^n\cos(n\theta) \\ y(\rho,\theta)=\sum_{n=1}^{\infty} c_n\rho^n\sin(n\theta) \end{cases} \quad (10)$$

where ρ and θ are module and phase of random vector in Fig.2(a).

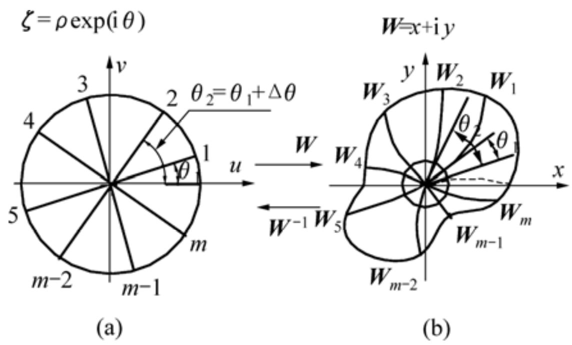

Fig.2 Conformal mapping between complicated region(a) and unit dish(b)

As shown in Fig.2, suppose boundary points $\boldsymbol{W}_k=x_k+\mathrm{i}y_k$ of region $\boldsymbol{W}$ are interpolation points, $k\in N$ (positive integer), and x_k and y_k are real part value and imaginary value of $\boldsymbol{W}_k$. By mapping function $\boldsymbol{W}$, mapping points of complex vectors $\boldsymbol{W}_1$, $\boldsymbol{W}_2, \cdots, \boldsymbol{W}_m$ are 1, 2, ⋯, m respectively as shown in Fig.2(a). Assuming that the phases $\theta_k\in[0, 2\pi]$ of complex vectors 1, 2, ⋯, m are arithmetical progression, then put $\boldsymbol{W}_1$, $\boldsymbol{W}_2$, ⋯, $\boldsymbol{W}_m$ coordinate points (interpolation points) value into Eqn.(10), when m is large enough, the infinite interpolation points in Eqn.(10) can be replaced by limited interpolation points. From the orthogonal character of trigonometric function, the values of real part a_j and imaginary part b_j of complex coefficients c_n in progression function of limit $m\in N$ can be calculated

$$\begin{cases} a_j=\dfrac{1}{m}\sum\limits_{k=1}^{m}\left[x_k\cos(j\theta_k)+y_k\sin(j\theta_k)\right] \\ b_j=\dfrac{1}{m}\sum\limits_{k=1}^{m}\left[-x_k\sin(j\theta_k)+y_k\cos(j\theta_k)\right] \end{cases} \tag{11}$$

In Eqn.(11), interpolation points $\boldsymbol{W}_1$, $\boldsymbol{W}_2$, ⋯, $\boldsymbol{W}_m$ are unknown satisfying boundary mapping conditions. The interpolation points can be defined as odd interpolation points when the interpolation points cannot satisfy calculating precision requirements. Adding new interpolation points between odd interpolation points, namely even points, then doing calculation of mutual iterative for odd and even interpolation points, and the interpolation points value and c_n value can be calculated under the condition of satisfying precision requirements.

4 Analysis of Ω of rectangle-plate region

There are two symmetrical axes in the rectangle-plates region, it can be proved that the coefficients in Eqn.(8) are real numbers, and the coefficients of even terms are zero. Then Eqn.(8) can be simplified as follows[15]:

$$\boldsymbol{W}=\sum_{k=1}^{\mathrm{N}}a_k\zeta^{2k-1} \tag{12}$$

As shown in Fig.1, suppose that the length and width of the simple-supported rectangle-plate are 2a and 2b, separately, and the arc radius is R, from the symmetry character, geometrical boundary function can be described in the first quadrant region:

$$\begin{cases} y=\dfrac{b}{2}, \quad 0\leqslant x<\dfrac{a}{2}-R \\ y=\sqrt{R^2-\left[x-\left(\dfrac{a}{2}-R\right)\right]^2}+\left(\dfrac{b}{2}-R\right), \quad \dfrac{a}{2}-R\leqslant x<\dfrac{b}{2} \\ x=\dfrac{a}{2}, \quad 0\leqslant y<\dfrac{b}{2}-R \end{cases} \tag{13}$$

As shown in Fig.(3), for rectangle-shape region with arc on boundary, by convergence method along normal direction, we can find out the interpolation points value of conformal mapping. Assuming that the circular centre of arc is (x_0, y_0), $z_k(x_k, y_k)$ is approximate interpolation point that approaches to arc z along normal direction, and R_k represents the distance away from (x_0, y_0), then the circular normal function is

$$\begin{cases} x-x_0=\dfrac{R}{R_k}(x_k-x_0) \\ y-y_0=\dfrac{R}{R_k}(y_k-y_0) \end{cases} \tag{14}$$

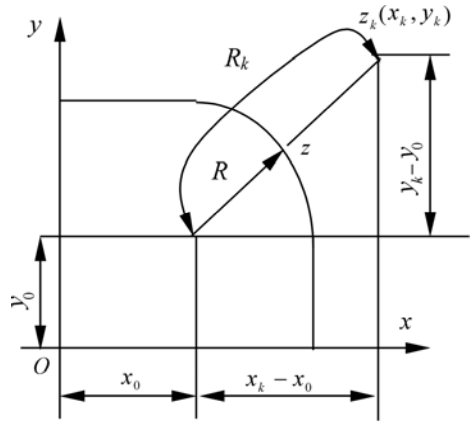

Fig.3 Arc convergence along normal direction

Under the given convergence precision by $1-R/R_1=eps$, z_1 can be converged in z along normal direction, whilst by mutual iterative between odd and even interpolation points z_k, the values of x_k and y_k in Eqn.(11) can be calculated, which is named as trigonometric interpolation method. By this method, both calculating precision of interpolation points and the coefficient c_n can be guaranteed.

As shown in Fig.1, when the two rectangle regions a=20, b=20 and a=26, b=15.38 with arc radius R=5, since there are no less than two symmetrical axes in the plate region, its conformal mapping function is Eqn.(12), over trigonometric interpolation iterative calculation (taking 16 points in the first quadrant to present infinite points in Eqn.(1)), both the coordinates of interpolation points and a_n can be computed.

x_k={13.0, 13.0, 12.783291, 11.216461, 8.983152, 7.309 824, 6.178 737, 5.260 732, 4.481374, 3.787 931, 3.158 911, 2.574 776, 2.024 826, 1.498 992, 0.990 405, 0.492 544, 0.0};

y_k={0.0, 1.811 632, 4.148 372, 6.520 411, 7.594 697, 7.692 308, 7.692 308, 7.692 308, 7.692 308, 7.692 308, 7.692 308, 7.692 308, 7.692 308, 7.692 308, 7.692 308, 7.692 308, 7.692 308};

a_{2i-1}={9.576 007, 2.653 134, 1.040 861, 0.306 351, −0.047 321, −0.186 868, −0.199 647, −0.143 173, −0.069 587, −0.018 722, 0.004 075, 0.017 783, 0.026 439, 0.013 211, −0.015 354, −0.001 398}

x_k={10.0, 10.0, 10.0, 10.0, 10.0, 10.0, 9.981 622, 9.581 253, 8.535 456, 7.002 865, 5.427 878, 4.193 685, 3.221 291, 2.344 464, 1.535 751, 0.759 072, 0.0};

y_k={0.0, 0.759 080, 1.535 761, 2.344 483, 3.221 325, 4.193 760, 5.428 306, 7.003 027, 8.535 611, 9.581 325, 9.981 659, 10.0, 10.0, 10.0, 10.0, 10.0, 10.0};

a_{2i-1}={10.730 256, 0.0, −0.949 442, −0.0, 0.292 243, 0.0, −0.099 062, −0.0, 0.027 623, 0.0, −0.005 607, −0.0, 0.007 171, −0.0, −0.025 723, 0.0};

a_{2i}=0, i=1, ⋯, 16.

Bring a_{2i-1} and Eqn.(12) into Eqn.(4), and suppose Ω and S_p are the coefficients of fundamental frequency and constant in-plane stress, respectively, then they are defined as

$$\Omega^2 = \frac{\rho_r h A^2}{G}\omega^2 , \quad S_p = \frac{A}{G} S \tag{15}$$

Assuming N=1 in Eqn.(6), and we can get

$$\Omega^2 = \frac{16HA^2 + 2\pi A S_p}{\left\{ \pi \sum_{j=1}^{N} \left[a_j^2 \frac{2j-1}{j(2j+1)} \right] + \lambda A g\left(\rho_p, \theta_p\right) \right\}} \tag{16}$$

where

$$g\left(\rho_p, \theta_p\right) = \left(1-\rho_p^2\right)^2 \left\{ \left[a_1 + \sum_{j=2}^{N} a_j (2j-1) \rho_p^{2j-2} \cos(2j-2)\theta_p \right]^2 + \left[\sum_{j=2}^{N} a_j (2j-1) \rho_p^{2j-2} \sin(2j-2)\theta_p \right]^2 \right\};$$

$$H = \iint_A \left(\frac{\partial^2 D}{\partial \zeta \partial \bar{\zeta}} \right)^2 \frac{1}{\|W'\|} \mathrm{d}u\mathrm{d}v.$$

Assuming that the plate with arc radius R=5, and a=24, b=16.667, when there is concentrated substance m_p at the point $p(x_p, y_p)$=(0, 0), doing calculation with Eqn.(4), the results are shown in Fig.4. For different constant in-plane stresses, the coefficient Ω of fundamental frequency reduces with the increase of concentrated mass, and increases with the enlargement of S_p, and the fundamental frequency of elastic plate in positive stress state is larger than that in the state of pressing stress.

For the same area size of elastic plates, different geometrical shapes will cause the change of fundamental frequency. As shown in Fig.5, when there is concentrated substance at p(0, 0), the coefficient Ω of fundamental frequency will reduce with the increase of ratio of length to width $e=a/b$.

For the rectangle region of a=20 and b=10 with arc radius R=5, Fig.(6) shows the change of Ω in a certain point with the concentrated substance $p(x_p, y_p)_{\rho\leqslant 1}$, Ω is the maximum when point p exists at θ=π/2 and θ=3π/2, and minimum Ω exists at θ=0 and θ=π. Whilst, the changing cycle is π, and which is consistent with the symmetrical character of the plates region. For the square shape region of a=20 and b=10, Ω is the maximum when point p exists at θ=0, θ=π/2, θ=π and θ=3π/2, and minimum Ω exists at θ=π/4, θ=3π/4, θ=5π/4 and θ=7π/4 respectively, the changing cycle is π/2, and this is consistent with four symmetrical axes of square-plates region.

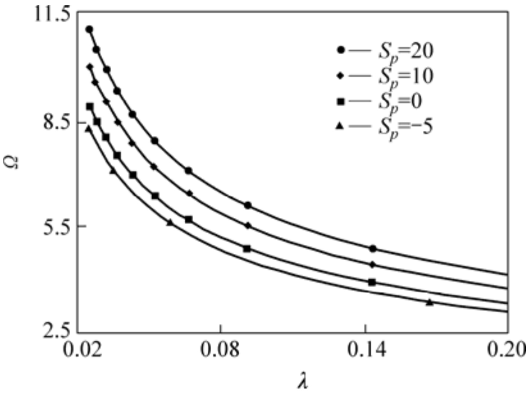

Fig.4 Ω varied with λ when a=24, b=16.667

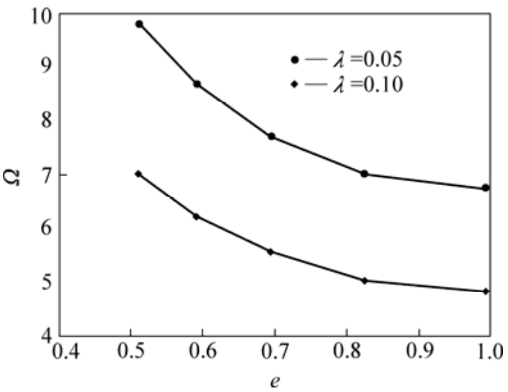

Fig.5 Ω varied with e when S_p=15

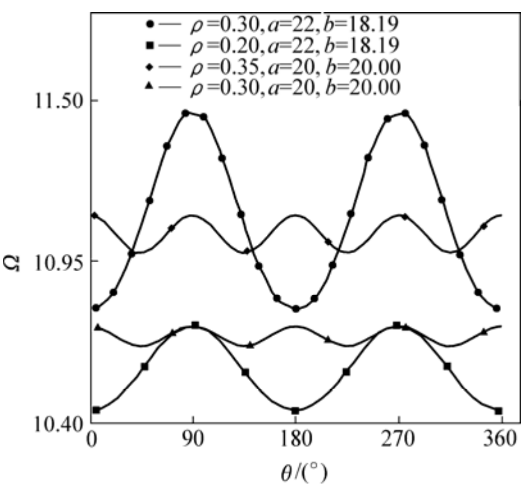

Fig.6 Ω varied with θ at S_p=20, λ=0.025

As shown in Fig.(7), with enlargement of the distance between $p(x_p, y_p)_{\theta=\theta_0}$ and (0, 0), fundamental frequency tends to increase. When p locates of (0, 0), Ω is the minimum, and the maximum exists in the outer boundary of plates. Whilst, when ρ=0 and ρ=1, Ω has the sole value, and there is no relationship with vector phase θ, because the value ρ corresponding to point p sits at (0, 0) and the outer boundary that satisfies simple-supported conditions.

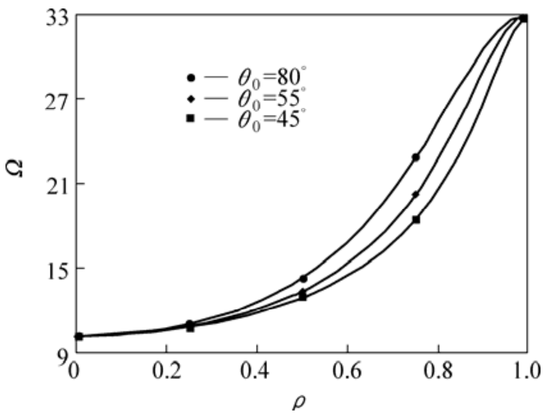

Fig.7 Ω varied with ρ when S_p=20, λ=0.025; a=22, b=18.18

5 Conclusions

1) Conformal mapping method of trigonometric interpolation was provided, and conformal mapping function that can realize the mutual conversion between rectangle-plates region with arc radius and unit dish region was carried out, as well as the vibrating function of complicated simple-supported elastic plate with concentrated substance was depicted by simple region under the in-plane state of constant stress. As the result, the fundamental frequency coefficient of the plate was calculated.

2) Coefficient of fundamental frequency Ω reduces with the increase of ratio of length to width e and λ the coefficient of substance ratio, and enlarges with the increase of ρ the module of random vector and constant in-plane stress S.

References

[1] WANG D J. Natural vibration of repetitive structure[J]. The Chinese Journal of Mechanics, 2000, 16(2): 85−95.

[2] CLOUH R W. Dynamics of Structure[M]. New York: McGraw-Hill Inc, 1993.

[3] KAPLUNVO J D. A Perturbation approach evaluating natural frequencies of moderately thick elliptic plates[J]. Journal of Sound and Vibration, 2005, 281: 905−919.

[4] WANG Fang. The limit analysis of irregular reinforced concrete slabs[J]. Journal of Central South University of Technology: Natural Science, 2001, 32(6): 573−576. (in Chinese)

[5] PAN Jun-qi. Recycling process assessment of mechanical recycling of circuit board[J]. Journal of Central South University of Technology, 2005, 12(S2): 157−161.

[6] TEPPATI C. Conformal-mapping design tools for coaxial couplers with complex cross section[J]. IEEE Transactions on Microwave Theory and Techniques, 2002, 50(10): 2339−2345.

[7] RAJALINGHAM C. Axisymmetric vibration of circular plates and its analog in elliptical plates using characteristic orthogonal polynomials[J]. Journal of Sound and Vibration, 1993, 161: 109−118.

[8] AMATORE C. Simulation of the double hemicylinder generator-collector assembly through conformal mapping technique[J]. Journal of Electro-analytical Chemistry, 2003, 553(S): 49−61.

[9] WU Guo-chuan. Tandem Blade Cascade[M]. Beijing: National Defense Industry Press, 1996. (in Chinese)

[10] ZOU Ji-bin. Magnetic Circuit and Magnetic Field[M]. Harbin: Harbin Industry University Press, 1998. (in Chinese)

[11] CRANDALL S. Engineering Analysis[M]. New York: McGraw-Hill Book Co, 1950.

[12] QI Hong-yuan, ZHU Heng-jun. Conformal mapping modeling of metal plastic deformation and die cavity in special-shaped extrusion[J]. Trans Nonferrous Metal Soc China, 2002, 12(5): 858−861.

[13] LAURA P A A. Fundamental frequency of vibration of clamped plates of arbitrary shape subjected to hydrostatic state of in-plane stress[J]. Journal of Sound and Vibration, 1976, 48(3): 327−332.

[14] NETKU Y. Conformal Field Theory[M]. Cambridge, Massachusetts USA: Perseus Pub, 2000.

[15] QI Hong-yuan, ZHU Heng-jun. Analysis of metal forming and 3D die cavity in special/shaped extrusion[C]// Proceeding of Chinese Postdoctoral Academic Conference. Beijing: Science Press, 2001: 484−488.

(Edited by YANG Hua)